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Probability Methods and Nonlinear Analysis

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The concepts of accretive and differentiable operator in a Banach space B are used to show that certain approximations to a solution of a nonlinear evolution equation converge. When B is a space of continuous functions it is shown that the approximations and the solution be represented as integrals with respect to a signed measure on a function space. As an example, a new proof is given for the existence and uniqueness of solutions to a nonlinear parabolic differential equations with coefficients dependent upon solutions. Integral representations of these solutions follow.

1. INTRODUCTION

From the point of view of topological measure theory and functional analysis, some of the fundamental methods of the theory of Markov processes need not be limited to the study of probability measures. For example, the Kolmogorov type consistency theorem for projective systems of measures, the Prokhorov tightness condition for compactness of a set of measures, the compactness argument for proving path continuity, and the concepts of transition operators and transition probabilities, all have generalizations which apply to signed measures. The well known connections between Markov processes and the study of second order elliptic operators (cf. [8, 5, 13, 1, 9, and 19]), can also be extended in some ways to connections between more general linear elliptic operators and signed measures on function spaces. If one can tolerate the present lack of physical interpretation, such signed measures may be thought of as (stochastic) processes whose sets of trajectories are measured by a signed measure.

The relationship with nonlinear analysis is less straightforward. The approach of the present article relies upon two important concepts which are familiar to nonlinear analysis. The first is the differential of a nonlinear operator. Approximation of a nonlinear operator by this linear operator makes available the many tools of linear functional analysis (cf. [18, 16, and 15]). If the operator is of elliptic type then the derivative is possibly associated with a measure on a function space as indicated above. The second concept is a generalization of ellipticity, a close relative of the accretive or monotone operator as studied for example in [17], [14], [11], [10], [2], [4], and [3]. This concept is especially useful in showing that an approximation to the solution of a differential equation must converge.

After developing the measure theoretical generalizations of probabilistic results in Section 2 we show in Section 3 that certain conditions of differentiability and accretiveness can be used in a new and simple way prove the convergence of certain approximations to the solution of a nonlinear evolution equation. The approximations are given in terms of the fundamental solution to the evolution equation involving the derivative of the nonlinear operator. In Section 4 we show that the approximations are naturally associated with signed measures on a function space. It is natural to adopt the concepts and terminology of stochastic processes. A solution and its approximations are represented as integral over a space consisting of continuous paths of a "signed" process. In Section 5 we look at a basic example which may be thought of as a diffusion equation with coefficients dependent upon the solution. Solutions to such equations are well known to exist (cf. [6 and 12]), and may lead to a diffusion theory when the solution is known. The result of the present article is a new method of proof of existence and uniqueness of solutions with connections to a different but related class of processes whose study seems promising.

2. Some Generalizations of Probability Theorems to Signed Measure Spaces

We now gather some definitions and results which are needed in Section 4. The proofs of these propositions by arguments involving compactness and continuous function spaces are not far from standard and are found in [7]. The section closes with a useful lemma.

The first result is a generalization of the Kolmogorov consistency theorem, the basic existence theorem for stochastic processes. Suppose that I is a directed index set and (X_i, g_{ij}, μ_i, I) is a projective system of Compact Hausdorff spaces X_i, continuous linear maps g_{ij} from X_j onto X_i $(i \leqslant j)$ and signed regular Borel measures μ_i on X_i. This means that $g_{ij}\mu_j = \mu_i$ and $g_{ij} \circ g_{jk} = g_{ik}$ when $i \leqslant j \leqslant k$. Let

(g_i) be a consistent family of continuous maps of a compact Hausdorff space Y onto X_i ($g_i = g_{ij} \circ g_j$) which separate the points of Y.

2.1 PROPOSITION. *If $\| \mu_i \| \leqslant C$ for each i then there is a unique signed regular Borel measure μ on Y such that $\| \mu \| \leqslant C$ and $g_i \mu = \mu_i$. If $\mu_i \geqslant 0$ for each i then $\mu \geqslant 0$, and if each μ_i is a probability measure, then so is μ. (For proof see* [7], *Prop.* 6.5).

The standard method for proving the existence of Markov processes is to define a consistent family of measures on the finite dimensional projections of a product space. The measures are obtained from a transition probability which may arise from a Markov semigroup of bounded linear operators. Slightly more general is the concept of transition operator. Let B be a Banach space and let I be an interval of real numbers. For $s \leqslant t$ in I suppose that T_{ts} is a bounded linear operator on B. If $T_{ss} = 1$ and $T_{ts} \circ T_{sr} = T_{tr}$ for $r \leqslant s \leqslant t$ then T is called a *transition operator* on B indexed by I. If T_{ts} depends only on $t - s$ then T is called a semigroup of bounded linear operators. If T is a transition operator on the space $C(X)$ of continuous functions on a compact Hausdorff space X then T_{ts} defines a family $T(t,s, x, dy)$ of measures on X by the formula $\int f(y)\, T(t, s, x, dy) = T_{ts} f(x)$ for each f in $C(X)$, x in X, and $s \leqslant t$. The following proposition shows how a transition operator defines a signed measure on a product space. Let I be the collection of finite subsets of the interval $[a, b]$ directed by inclusion. Let g_i and g_{ij} be the natural projections of $Y = X^{[a,b]}$ onto X^i and of X^j onto X^i where $i \subseteq j$.

2.2 PROPOSITION. *Let X be a compact Hausdorff space and T a transition operator on $C(X)$ indexed by $[a, b]$ such that $\| T_{ts} f \| \leqslant \| f \|$ $\exp[c(t - s)]$ for each f, s, t. Suppose that ν is a signed regular Borel measure on X and for each finite subset $i = \{t_0, \ldots, t_n\}$ of $[a, b] = [t_0, t_n]$ and for each f in $C(X^i)$ let*

$$\mu_i(f) = \int \nu(dx_n) \int T(t_n, t_{n-1}, x_n, dx_{n-1}) \times \int \cdots \int T(t_1, t_0, x_1, dx_0) f(x_0, \ldots, x_n),$$

then there is a unique signed measure μ on $X^{[a,b]}$ such that $g_i \mu = \mu_i$ for each i and $\| \mu \| \leqslant \exp[c(b - a)]$. (For proof see [7], *Prop.* 8.1).

2.3 DEFINITION. Suppose that X is a topological space, A is a nonempty set, and $(\Omega, \mathscr{F}, \mu)$ is a signed measure space. An X-valued signed process indexed by A over $(\Omega, \mathscr{F}, \mu)$ is a mapping x of Ω into X^A such that for each t in A, the function $w \to (x(w))(t) = x(t, w)$ is Borel measurable.

A signed measure on X^A determines a signed process, namely the identity map, and a signed process x determines a signed measure $x(\mu)$ on X^A. We adopt the terminology of the theory of stochastic processes, for example the map $t \to x(t, w)$ is called a path or trajectory, and in Proposition 2.2 we write $g_i\mu(f) = \int f(x(w, t_0),..., x(w, t_n))\, \mu(dw)$. If A is a topological space, one important question is whether, for a given signed process x, almost all trajectories are continuous, or equivalently, whether $x(\mu)$ is concentrated on the space of continuous functions from A into X. The following proposition gives an answer.

2.4 Proposition. *Suppose that I is an interval in R, X is the one-point compactification of R^n, and μ is a signed regular Borel measure on X^I. If there are positive numbers C, d, and $p > 1$ such that for s, t in I,*

$$\int | w(s) - w(t)|^p \mid \mu \mid(dw) \leqslant C \mid t - s \mid^{d+1}, \tag{1}$$

then μ is concentrated on $C(I, R^n)$ the space of continuous functions from I into R^n, and further, μ defines a signed regular Borel measure on the metric space $C(I, R^n)$ with the topology of uniform convergence on I.

The proof of this proposition is based on Prokhorov's tightness conditions as is the following (for proofs see [7, Props. 11.2.2, 11.2.3, and 12.2].)

2.5 Proposition. *If X and I are as in 2.4 and (μ_a) is a family of signed regular Borel measures on $C(I, R^n)$ such that* (i) *$sup\{| \mu_a |(X)\} < \infty$,* (ii) *for given $\epsilon > 0$ there is an m such that $| \mu_a | \{w: | w(0)| \geqslant m\} < \epsilon$ for each a, and* (iii) *there are positive numbers C, d, and $p > 1$ such that for s, t in I and for each a*

$$\int | w(s) - w(t)|^p \mid \mu_a \mid(dw) \leqslant C \mid t - s \mid^{d+1}.$$

Then the family (μ_a) is relatively compact in the space M of signed *regular Borel measures on $C(I, R^n)$ with the topology $\sigma(M, C)$ where C is the space of bounded continuous functions on $C(I, R^n)$.*

Suppose that X is the one point compactification of R^n. A transition operator P on $C(X)$ such that $\| P_{ts} \| = 1$ for each s and t and $P_{ts} f \geqslant 0$ whenever $f \geqslant 0$, is called a Markov transition operator. For a Markov transition operator the hypotheses of proposition 2.2 are satisfied, the measure on the product space so determined is a probability measure, and the corresponding process is a Markov process. The classical example of a Markov process is Brownian motion whose transition operator is given by $P_{ts} f(x) = \int g(x, y, t, s) \; f(y) \; dy$ where $g(x, y, t, s) = [4\pi(t - s)]^{-n/2} \times \exp[-c \mid x - y \mid^2/4(t - s)]$. For this transition operator the equation (1) of proposition 2.4 is implied by the inequality $\int \mid x - y \mid^4 g(x, y, t, s) \, dy \leqslant C_0 \mid t - s \mid^2$; hence almost all trajectories of Brownian motion are continuous, R^n valued functions. For proving path continuity of more general signed processes and for proving compactness of families of signed measures we refer to the following definition and lemma.

2.6 Definition. Suppose that T is a transition operator on $C(X)$ indexed by $A = [a, b]$. We say that T satisfies a condition $P(M, c)$ if there is a Markov transition operator P on $C(X)$ indexed by A such that $P(t, s, x, dy) \leqslant Mg(t, s, x, y) \, dy$ and $\mid T(t, s, x, dy) \mid \leqslant P(t, s, x, dy) \exp(c(t - s))$ where M and c are positive constants.

Note that if T satisfies condition $P(M, c)$ then $\| T_{ts} \| \leqslant \exp[c(t - s)]$ so that by proposition 2.2, a measure on X^A is determined for each initial measure ν.

2.7 Lemma. *If T is a transition operator on $C(X)$ indexed by $A = [a, b]$ and if T satisfies condition $P(M, c)$ then for an initial measure ν on R^n, T determines a measure μ on X^A which satisfies*

$$\int \mid w(s) - w(t) \mid^4 \mid \mu \mid (dw) \leqslant C \mid t - s \mid^2 \text{ where } C \leqslant 2c_0 M \, \| \nu \| \exp(c(b - a)).$$

Proof. Let μ be the measure determined by T as in proposition 2.2. Let $f(x, y) = \mid x - y \mid^4$; then $\int \mid w(s) - w(t) \mid^4 \mid \mu \mid (dw) = \int f(w(s), w(t)) \mid \mu \mid (dw) = \mid \mu \mid (f)$. Also $\mid \mu \mid (f) \leqslant 2 \sup\{\mid \mu(h) \mid : 0 \leqslant h \leqslant f\} = 2 \sup\{\mid \mu_i(h) \mid : 0 \leqslant h \leqslant f, \{0, t, s, a\} \subseteq i, \; h \in C(X^i)\}$. Now if $i = \{t_0, ..., t_m\}$ contains $\{0, t, s, a\}$, h is in $C(X^i)$, and $0 \leqslant h \, (w(t_0), ..., w(t_m)) \leqslant f(w(t), w(s))$ for each w, then

$$
\begin{aligned}
|\mu_i(h)| &\leqslant \int |\nu|(dx_m) \int |T(t_m, t_{m-1}, x_m, dx_{m-1})| \int \cdots, \\
&\qquad \times \int |T(t_1, t_0, x_1, dx_0)|\, h(x_0, \ldots, x_m) \\
&\leqslant \exp(c(b-a)) \int |\nu|(dx_m) \int P(t_m, t_{m-1}, x_m, dx_{m-1}) \int \cdots, \\
&\qquad \times \int P(t_1, t_0, x_1, dx_0)\, f(x_s, x_t) \\
&\leqslant \exp(c(b-a)) \int |\nu|(dx_m) \int P(t, s, x_t, dx_s)\, f(x_s, x_t) \\
&\leqslant M \|\nu\| \exp(c(b-a)) \int g(t, s, x, y)\, |x-y|^4\, dy \\
&\leqslant C_0\, |t-s|^2\, M \|\nu\| \exp(c(b-a)).
\end{aligned}
$$

3. Evolution Equations for K-Accretive, G-Differentiable Operators

Let B be a Banach space and let D be a dense linear subspace of D.

3.1 Definition. An operator A from a subset of D into B is said to be *G-differentiable* whenever for each u in the domain of A there is a linear operator $dA(u)$ from D into B such that if $f(u, w) = A(u + w) - Au - dA(u)(w)$ then for each w in D, $(1/t) f(u, tw) \to 0$ as $t \to 0$. We say that a mapping u from an interval $[0, a]$ into D is consistent if $(1/s) f(u(t), u(t + s) - u(t)) \to 0$ uniformly in t as $s \to 0$.

Throughout this section we assume that A is a nonlinear operator from D into B. We are concerned with solutions $u(t)$ for the equation

$$D_t u(t) = Au(t) \tag{1}$$

on an interval $[0, a]$ for a given initial condition $u(0) = u_0$. First we look at an approximation of a given solution based on the Hille–Yosida theory of semigroups of bounded linear operators generated by the differential $dA(u)$. The following is simple to prove.

3.2 Lemma. *If A is G-differentiable, if $u(t)$ is a consistent solution to* (1) *on the interval* $[0, a]$, *if $dA(u(t))$ is closeable* (*see* [20]) *and $A\, u(t)$ is differentiable in t then $D_t^2(u(t)) = D_t A\, u(t) = \overline{d\, A(u(t))} A\, u(t)$.*

According to the Hille–Yosida theorem [20], if the domain of the resolvent $(1 - r\, dA(u))^{-1}$ is all of B and if $\|(1 - r\, dA(u))^{-1}\| \leqslant$

$(1 - rK)^{-1}$ for some positive constant K. Then $dA(u)$ is the infinitesimal generator of a semigroup $S_t(u)$ of bounded linear operators such that $\| S_t(u)\| \leqslant \exp(tK)$. If this is the case we can form an approximation to a given solution of (1) as follows. For given n, we subdivide the interval $[0, a]$ into intervals of equal length $r = a/2^n$. Let $u_n(0) = u_0$, $u_n(t) = u_0 + tS_t(u_0)\, Au_0$ for $0 \leqslant t \leqslant r$ and let $u_n(kr + t) - u_n(kr) = tS_t(u(kr))\,(u_n(kr) - u_n(kr - r)) = tS_t(u(kr)) \times \prod_{j=1}^{k} S_r(u(kr - jr))\, Au_0 = t \prod (n, kr + t)$ for $1 \leqslant k \leqslant 2^n$ and $0 \leqslant t \leqslant r$. Recall that a function w from $[0, a]$ into B is *uniformly differentiable* if and only if $(1/h)\,(w(t + h) - w(t) - D_t\, w(t)) \to 0$ uniformly in t as $h \to 0$. The proof of the following proposition is given in [7].

3.3 PROPOSITION. *If A is G-differentiable, $u(t)$ is a consistent solution to* (1), *$Au(t)$ is uniformly differentiable, and $dAu(s)$ is the infinitesimal generator of a semigroup $S_t(u(s))$ such that* $\| S_t(u(s))\| \leqslant \exp(tK)$ *and* $(S_t(u(s)) - 1)\, dA(u(s))\, Au(s) \to 0$ *uniformly for s in* $[0, a]$, *then $u_n(t) \to u(t)$ and $\prod (n, t) \to Au(t)$ uniformly for t in* $[0, a]$.

Thus we may be led to think of a representation of the solution $u(t)$ as

$$u(t) = u_0 + \int_0^t ds \prod_{r=0}^{s} S_{dr} \quad (u(r))\, Au_0 .$$

Unfortunately the abstract theory of linear semigroup is not rich enough to prove applicable existence theorems based on this type of representation. We must replace the semigroup with a time dependent fundamental solution to the linear evolution equation $D_t w(t) = dA(u(t))\, w(t)$. (See [7] for the semigroup approach.)

3.4 DEFINITION. Suppose that A is G-differentiable. A family E of functions from $[0, a]$ into D is called a generating class for dA on $[0, a]$ whenever the following conditions are satisfied.

(a) If u is in E and u^t is defined by $u^t(s) = u(s)$ for $s \leqslant t$ and $u^t(s) = u(t)$ for $s \geqslant t$ then u^t is in E.

(b) For each u in E there is a transition operator $T(u, t, s)$ on B indexed by $[0, a]$ such that $T(u, t, s)\, w$ is in D for each w in B and $D_t\, T(u, t, s)w = dA(u(t))\, T(u, t, s)w$.

(c) For each w in D the family $\{D_t T(u, t, s)w : u \text{ in } E\}$ is uniformly bounded and uniformly equicontinuous in t.

(d) For each w in D, if $w(s+t) = T(u(s), t, 0)\ T(u, s, 0)w = T(u^s, s+t, 0)w$ then $(1/t) f(u(s), tw(s+t))$ converges to 0 uniformly in u and s as t goes to 0.

Note that if u is constant then $T(u, t, s) = T(u, t-s)$ is a semigroup of bounded linear operators whose infinitesimal generator is $dA(u)$. In this case we have $D_t T(u, t)w = dA(u)\ T(u, t)w = T(u, t)\ dA(u)w$. Condition ($c$) of the preceeding definition implies that $(T(u(s), t) - 1) \times dA(u(s))\ T(u, s, 0)w$ converges to 0 uniformly for u in E and for $0 \leqslant s < a$ as t goes to 0.

3.5 Definition. Let K be a nonnegative real valued function on D. The operator A is said to be *K-accretive* if for each u and v in D and positive r,

$$\| u - v + rAu - rAv \| \geqslant (1 - rK(u)) \| u - v \|.$$

The following propositions are easy to prove.

3.6 Proposition. *If A is K-accretive then $1 + rA$ is one to one on $\{u : rK(u) < 1\}$. If f and g are in the range of A then*

$$\|(1 + rA)^{-1} f - (1 + rA)^{-1} g \| \leqslant (1 - rJ)^{-1} \| f - g \|$$

where $J = K((1 - rA)^{-1} f)$.

3.7 Proposition. *If $-A$ is G-differentiable and K-accretive then $-dA(u)$ is $K(u)$-accretive for each u in D, hence on the domain of the resolvent of $dA(u)$ we have $\|(1 - rdA(u))^{-1}\| \leqslant (1 - K(u))^{-1}$.*

3.8 Proposition. *If $-A$ is K-accretive and G-differentiable, u is in the generating class for dA on $[0, b]$ and $\| Ku(t)\| \leqslant C$ for each t then for each w in B, $\| T(u, t, s)w \| \leqslant \| w \| \exp C(t - s)$.*

Proof. Consider a subdivision of the interval $[s, t]$ into intervals of length $h = (t-s)/n$. If $w(r) = T(u, r, s)w$ then $w(r+h) - w(r) = hdA(u(r+h))\ w(r+h) + \epsilon(r, h)$ where $(1/h)\ \epsilon(r, h) \to 0$ uniformly in r. It follows that $w(r+h) = [1 - h\ dA(u(r+h))]^{-1}(w(r) + \epsilon(r, h))$. Since $-dA(u(r+h))$ is $K(u(r+h))$-accretive we have $\| w(r+h)\| \leqslant (1 - hC)^{-1} (\| w(r)\| + \| \epsilon(r, h)\|)$. It follows that

$$\| w(t)\| \leqslant (1 - hC)^{-n} \| w \| + (t - s)(1 - hC)^{-n} \max \| \epsilon(t_k, h)/h \|$$

The conclusion is obtained by letting n go to infinity.

3.9 DEFINITION. Suppose A maps D into B and u_0 is in D. We say that A and u_0 satisfy the approximation conditions on $[0, a]$ whenever

(1) $-A$ is G-differentiable and K-accretive.

(2) $A\, u_0$ is in D.

(3) There is a generating class E for dA on $[0, a]$ such that u_0 is in E and with the following property.

For each positive integer n, if $[0, a]$ is subdivided into intervals of length $r = a/2^n$, if

$$u_n{}^1(t) = u_0 + tT(u_0, t)\, Au_0 \quad \text{and if for each} \quad k \leqslant 2^n,$$
$$u_n{}^k(t) = u_n^{k-1}(t) \quad \text{for} \quad t \leqslant kr \quad \text{and,}$$
$$u_n{}^k(kr + t) = u_n^{k-1}(kr) + tT(u_n^{k-1}(kr), t)\, T(u_n^{k-1}, kr, 0)\, Au_0\,,$$

then for each k, $u_n{}^k$ is in E. (we denote $u_n{}^p = u_n$ when $p = 2^n$ so that for each k, $u_n^{k-1}(t) = u_n(t)$ for $t \leqslant kr$).

(4) There is a constant C such that $\| K(u_n(t))\| \leqslant C$ for each n and t.

3.10 LEMMA. *If A and u_0 satisfy the approximation conditions and if for $r = a/2^n$, $0 \leqslant k < 2^n$, and $0 < t \leqslant r$,*

$$b(n, k, t) = T(u_n(kr), t)\, T(u_n, kr, 0)\, Au_0 - Au_n(kr + t),$$

then $b(n, k, t) \to 0$ uniformly in $s = kr + t$ as n increases.

Proof. Let $c(n, k) = T(u_n, kr, 0)\, Au_0 - Au_n(kr)$.
By adding and subtracting $c(n, k)$ we get

$$\begin{aligned} b(n, k, t) &= (T(u_n(kr), t) - 1)\, T(u_n, kr, 0)\, Au_0 + c(n, k) \\ &\qquad - tdA(u_n(kr))\, T(u_n(kr), t)\, T(u_n, kr, 0)\, Au_0 - f(n, k, t) \\ &= c(n, k) - f(n, k, t) \\ &\qquad + \int_0^t ds(T(u_n(kr), t) - T(u_n(kr), s))\, dA(u_n(kr))\, T(u_n, kr, 0)\, Au_0, \end{aligned}$$

where $f(n, k, t) = f(u_n(kr), tT(u_n(kr, t)\, T(u_n, kr, 0)\, Au_0)$.

Since the integrand is bounded and $f(n, k, t) \to 0$ we need only examine $c(n, k)$. Adding and subtracting $c(n, k - 1) + rdA(u_n(kr)) \times T(u_n, kr, 0)\, Au_0$ we obtain

$$c(n, k) = c(n, k - 1) - f(n, k - 1, r) + rq(n, k - 1) + rp(n, k - 1),$$

where

$$q(n, k-1) = (1 - T(u_n(kr), r))\, dA(u_n(kr))\, T(u_n, kr - r, 0)\, Au_0,$$

and

$$rp(n, k-1) = \int_{kr-r}^{kr} dt(D_tT(u_n, t, 0) - D_tT(u_n, kr-r, 0))\, Au_0.$$

It follows that

$$c(n, k) = c(n, 1) + \sum_{j=1}^{k-1} rp(n, k-j) + rq(n, k-j) - f(n, k-j, r),$$

where

$$c(n, 1) = -f(n, 0, r) + \int_0^r D_tT(u_n(kr), t)\, Au_0 - r\, dA(u_0)\, T(u_0, r)\, Au_0.$$

Hence $c(n, k) \to 0$ as n increases.

3.11 THEOREM. *If A and u_0 satisfy the approximation conditions if $u(t)$ is a uniformly differentiable solution to $D_tu(t) = Au(t)$ such that $u(0) = u_0$, then $u_n(t) \to u(t)$ uniformly on $[0, a]$.*

Proof. If $te(s, t) = u(s+t) - u(s) - tAu(s+t)$, then

$$(1 - tA)^{-1}(u(s) + te(s, t)) = u(s+t).$$

Since $t \cdot b(n, k, t) = u_n(kr + t) - u_n(kr) - tAu_n(kr + t)$ we also have $(1 - tA)^{-1}(u_n(kt) + tb(n, k, t)) = u_n(kr + t)$. It follows that

$$\begin{aligned}
&\|u_n(kr+t) - u(kr+t)\| \\
&\quad = \|(1 - tA)^{-1}(u_n(kr) + t \cdot b(n, k, t)) - (1 - tA)^{-1}(u(kr) + te(kr, t))\| \\
&\quad \leqslant (1 - tC)^{-1}\|u_n(kr) - u(kr)\| + t(1 - tC)^{-1}\|b(n, k, t) - e(kr, t)\| \\
&\quad \leqslant t(1 - tC)^{-1}\|b(n, k, t) - e(kr, t)\| \\
&\qquad + (1 - tC)^{-1}\sum_{j=1}^{k} r(1 - rC)^{-j}\|b(n, k-j) - e(kr - jr, r)\| \\
&\quad \leqslant (1 - tC)^{-1}(t + a(1 - aC/p)^{-p}\sup\|b(n, j) - e(jr, r)\|)
\end{aligned}$$

where $p = 2^n$. Since u is uniformly integrable $e(jr, r) \to 0$ uniformly and by lemma 3.10 $b(n, j) \to 0$ uniformly. It follows that $\|u_n(s) - u(s)\| \to 0$ uniformly in s.

3.12 LEMMA. *Suppose that A and u_0 satisfy the approximation conditions. Suppose that $n < m$, $s = a/2^m$, $r = ps = a/2^n$, $0 < l < p$ and $0 < h \leqslant s$.* If $h \cdot d(n, k, l, h) = u_n(kr + ls + h) - u_n(kr + ls) -$

$hA(u_n(kr + ls + h))$ then $d(n, k, l, h) \to 0$ uniformly in $kr + ls + h$ as n increases.

Proof. Since

$$u_n(kr + ls + h) = u_n(kr) + (ls + h)\, T(u_n(kr), ls + h)\, T(u_n , kr, 0)\, Au_0 ,$$

by adding and subtracting $ls\, T(u_n(kr), ls)\, T(u_n , kr, 0)\, Au_0$ we obtain

$$\begin{aligned} hd(n, k, l, h) &= ls(T(u_n(kr), h) - 1)\, T(u_n(kr), ls)\, T(u_n , kr, 0)\, Au_0 \\ &\quad + hT(u_n(kr), ls + h)\, T(u_n , kr, 0)\, Au_0 - hAu_n(kr + ls + h)\cdot \end{aligned}$$

Next we add and subtract $h \cdot c(n, k)$, (see proof of 3.10) to get

$$\begin{aligned} hd(n, k, l, h) &= hc(n, k) - hf(n, k, ls + h) \\ &\quad - h\, dA(u_n(kr))(ls + h)\, T(u_n(kr), ls + h)\, T(u_n , kr, 0)\, Au_0 \\ &\quad + ls(T(u_n(kr), h) - 1)\, T(u_n(kr), ls)\, T(u_n , kr, 0)\, Au_0 . \\ &\quad + h(T(u_n(kr), ls + h) - 1)\, T(u_n , kr, 0)\, Au_0 . \end{aligned}$$

Examining $d(n, k, l, h)$ term by term using 3.10, and 3.4 we see that each term converges to 0 uniformly in $ls + h$ as n increases.

3.13 THEOREM. *If A and u_0 satisfy the approximation conditions, then there is a function u from $[0, a]$ into B such that $u_n(t)$ converges to $u(t)$ uniformly. Further, there is a subsequence of the sequence $Au_n(t)$ which converges uniformly to a continuous function v from $[0, a]$ into B such that $u(t) = u_0 + \int_0^t v(s)\, ds$.*

Proof. Suppose that $n < m$, $s = a/2^m$, $r = ps = a/2^n$, $0 < l < p$ and $0 < h \leqslant s$. We compute

$$\begin{aligned} &\| u_n(kr + ls + h) - u_m(kr + ls + h)\| \\ &\quad = \|(1 - hA)^{-1} (u_n(kr + ls) + hd(n, k, l, h)) - (1 - hA)^{-1} (u_m(kr + ls) \\ &\qquad + hb(m, kp + l, h))\| \leqslant (1 - hC)^{-1} (\| u_n(kr + ls) - u_m(kr + ls)\| \\ &\qquad + \| hd(n, k, l, h) - hb(m, kp + l, h)\|) \\ &\quad \leqslant (1 - hC)^{-1} (\| hd(n, k, l, h) - hb(m, kp + l, h)\| \\ &\qquad + \sum_{j=1}^{l} s(1 - sC)^{-1} \| d(n, k, l - j, s) - b(m, kp + l - j, s)\| \\ &\qquad + (1 - sC)^{-l} \| u_n(kr) - u_m(kr)\|) \\ &\quad \leqslant (1 - hC)^{-1} \left(h + s \sum_{j=1}^{kp+l} (1 - sC)^{-j}\right) \max \| d(n, k, j, h) - b(m, j, h) \| \\ &\quad \leqslant (1 - hC)^{-1} \max \| d(n, k, j, h) - b(m, j, h)\| \cdot (h + s \textstyle\sum (1 - sC)^{-j}), \end{aligned}$$

which goes to 0 uniformly as n increases. Let $u(t)$ be the limit of the Cauchy sequence $u_n(t)$.

To prove the second statement let $w_n(t) = T(u_n, t, 0)\, Au_0$ for each n and t. By proposition 3.8 we have $\| w_n(t)\| \leqslant \| Au_0 \| \exp(Ca)$ for each n and t. By using condition (c) of Definition 3.4 and the inequality

$$\| w_n(t) - w_n(s)\| = \left\| \int_s^t D_h T(u_n, h, s)\, w_n(s)\, dh \right\| \leqslant (t - s) \sup \| D_h w_n(h)\|$$

we see that the family $\{w_n\}$ is equicontinuous. By Ascoli's theorem there is a subsequence which converges uniformly to a continuous function v from $[0, a]$ into B. For simplicity of notation let us assume that $w_n(t)$ converges to $v(t)$ uniformly in t. Define $v_n(t) = T(u_n(kr), t - kr)$ $w_n(kr)$ whenever $kr \leqslant t + r$ and $r = a/2^n$. Since $v_n(t) - w_n(t) =$ $(T(u_n(kr), r) - 1)\, w_n(kr)$ we see that $v_n(t)$ converges to $v(t)$ uniformly. Since $v_n(t) - Au_n(t) = b(n, k, t - kr)$, Lemma 3.10 implies that $Au_n(t)$ converges to $v(t)$ uniformly in t. Finally let $y_n(t) = T(u_n(kr), r)$ $w_n(kr)$ for $kr \leqslant t < kr < Kr + r$. Then $u_n(kr) = u_0 + \int_0^{kr} y_n(t)\, dt$ and $y_n(t)$ converges uniformly to $v(t)$. Since v is Bochner integrable and $\int_0^t | y_n(s) - v(s)|\, dt$ converges to 0, it follows that $\int_0^t y_n(s)\, ds$ converges to $\int_0^t v(s)\, ds$ (cf. [20] p. 138). Since $u_n(t)$ converges to $u(t)$ we have $u(t) = u_0 + \int_0^t v(s)\, ds$.

4. Integral Representation of Solutions and Approximations

In this section let X be the one point compactification of R^m and let $B = C(X)$. The space B can be identified with the Banach space generated by the constant functions and the continuous functions vanishing at infinity in R^m. Let D be a dense subspace of B and let A be a mapping of D into B and suppose that A and u_0 satisfy the approximation conditions of Section 3 on an interval $I = [0, a]$. We first look at the possibility of an integral representation of a given solution of the evolution equation

$$D_t u(t) = Au(t), \qquad \text{where} \quad u(0) = u_0 .$$

4.1 Theorem. *Suppose that u is a solution of* (1) *which is in a generating class E for dA on I. Suppose also that*

(a) *$dA(u(t))$ is closeable for each t.*

(b) *$Au(t)$ is differentiable in t and $Au(t)$ is in D.*

(c) *The solution $v(t)$ to the equation $D_t(t) = dA(u(t))\, v(t)$ is unique for the initial condition Au_0.*

(d) *$Ku(t)$ is bounded by a constant C.*

(e) *T satisfies a condition $P(M, c)$.*

Then for each x_0 in R^m and t in I there is an R^m-valued process $x(t)$ with continuous trajectories over a signed measure space $(W(t), F(t), m(t))$ such that $x(t)(0, w) = x_0$ for each w and $Au(t)(x_0) = \int Au_0(x(t)(t, w))\, m(t)(dw)$. Hence

$$u(t)(x_0) = u_0(x_0) + \int_0^t ds \int A\, u_0(x(s)(s, w))\, m(s)(dw).$$

Proof. Since u is in a generating class E, the transition operator $T(u, s, r)$ exists and since $\| K(u(s)\| \leqslant C$, proposition 3.8 says that $\| T(u, s, r)\| \leqslant \exp(C(s - r))$. Corresponding to the initial measure $\nu = \delta_{x_0}$ (the point mass), proposition 2.2 yields a unique signed measure $m(t)$ on X^I. The condition (e) together with Proposition 2.4 and Lemma 2.7 says that $m(t)$ may be considered as a regular Borel measure on $C([0, t], R^m) = W(t)$. Hence $m(t)$ defines an R^m-valued signed process $x(t)$ with continuous trajectories by the formula $x(t)(s, w) = w(t - s)$. Since $dA(u(s))$ is closeable and $Au(s)$ is differentiable and $Au(s)$ is in D Lemmas 3.2 implies that $D_s Au(s) = dA(u(s))\, Au(s)$. We also have $D_s T(u, s, 0)\, Au_0 = dA(u(s))\, T(u, s, 0) Au_0$ so that by the uniqueness (c) we have $Au(s) = T(u, s, 0)\, Au_0$. As in the remarks following propositions 2.1 and 2.2 we have

$$\begin{aligned} Au(s)(x_0) &= T(u, s, 0)\, Au_0(x_0), \\ &= \int T(u, s, 0, x_0, dx)\, Au_0(x), \\ &= \int Au_0(x(s)(w, s))\, m(s)(dw). \end{aligned}$$

The process $x(t)$ may be thought of as running backward with respect to the solution $u(t)$.

Recall the approximations u_n of Section 3. Let $T(n, t, s) = T(u_n(kr, t - kr)\, T(u_n, kr, s)$ for $r = a/2^n$, $s < kr \leqslant t$, $T(n, t, s) = T(u_n(kr), t - s)$ for $kr \leqslant s < t$, and $T(n, t, s) = T(u_n, t, s)$ for $s < t \leqslant kr$.

4.2 Theorem. *For each n, suppose that the transition operator $T(n, t, s)$ satisfies a condition $P(M, c)$ independent of n. If $r = a/2^n$, $kr \leqslant t < kr + r$, and x_0 is in R^m, then there is an R^m-valued process $x(n, t)$ indexed by $[0, t]$ with continuous paths over a signed measure space*

$(W(n, t), F(n, t), m(n, t))$ such that $x(n, t)(0, w) = x_0$ for each w in $W(n, t)$ and

$$T(n, t, 0)\, Au_0(x_0) = \int Au_0(x(n, t)(t, w))\, m(n, t)(dw).$$

It follows that

$$u_n(kr)(x_0) = u_0(x_0) + \sum_j r \int Au_0(x(n, jr)(jr, w))\, m(n, jr)(dw).$$

Proof. Since the approximation conditions are satisfied, $\| K(u_n(t))\| \leqslant C$ for each n and t; hence by Proposition 3.5, $\| T(n, t, s)\| \leqslant \exp(C(t - s))$ for each n, s, and t. Again by Proposition 2.2, corresponding to the initial measure δ_{x_0}, there is a unique signed measure $m(n, t)$ which has the required properties. Path continuity follows from Proposition 2.4, and Lemma 2.7.

4.3 THEOREM. *With the same hypothesis as Theorem* 4.2 *let u_n, u, and v be given as in Theorem* 3.13. *For each t, there is an R^m-valued process $x(t)$ with continuous paths indexed by $[0, t]$ over a space $(W(t), F(t), m(t))$ such that $x(t)(0, w) = x_0$ for each w in $W(t)$,*

$$v(t)(x_0) = \int Au_0(x(t)(t, w))\, m(t)(dw), \qquad \text{and}$$

$$u(t)(x_0) = u_0(x_0) + \int_0^t ds \int Au_0(x(s)(s, w))\, m(s)(dw).$$

Further, $v(t)(x_0)$ is the limit of a subsequence of the sequence

$$\int Au_0(x(n, t)(t, w))\, m(n, t)(dw) \qquad \text{of Theorem 4.2.}$$

Proof. As in Theorem 3.13 let $w_n(t) = T(u_n, t, 0)\, Au_0$ for each n and t. Again for convenience of notation we assume that the sequence $\{w_n\}$ converges uniformly to the function v. Since for each n and t we have $m(n, t)\, \{w\colon w(0) \neq x_0\} = 0$, $| m(n, t)|\, (X) \leqslant \exp(Ca)$, and by the condition $P(M, c)$ together with Lemma 2.7 we have $\int | w(s) - w(r)|^4 \, | m(n, t)|(dw) \leqslant M \mid s - r \mid^2$, it follows from Proposition 2.5 that there is a subsequence $m(n_k, t)$ which converges in the space of signed Randon measures on $C([0, a], R^m)$ to a measure $m(t)$. This means that for each continuous bounded function f on $C([0, a], R^m)$, $m(n, t)f$ converges to $m(t)f$. Since Au_0 is in B, the function $w \to Au_0(w(t))$ for w in $C([0, a], R^m)$ is continuous and bounded. It follows that $\int Au_0(w(t))\; m(n_k, t)(dw)$ converges to $\int Au_0(w(t))\; m(t)(dw)$. Since

$w_n(t) = \int Au_0(w(t))\, m(n, t)(dw)$ and $w_n(t)$ converges to $v(t)$ we have $v(t) = \int Au_0(w(t))\, m(t)(dw) = \int Au_0(x(t)(t, w))\, m(t)(dw)$.

5. EXAMPLE. Let B be the Banach space generated by the constants and the continuous functions on R^m which vanish at infinity, with the sup norm. Suppose that $0 < \alpha < 1$ and let $H^{2+\alpha}$ be the space of twice continuously differentiable functions whose derivatives of order $\leqslant 2$ are bounded and uniformly Holder continuous with exponent α. The space $H^{2+\alpha}$ is itself a Banach space with the norm

$$| u |^{2+\alpha} = \sum_{|k|=2} \langle D^k u\rangle_x^\alpha + \sum_{|k|\leqslant 2} \| D^k u \|$$

where $\langle\cdot\rangle_x^\alpha$ is the Hölder constant and $\|\cdot\|$ is the sup norm. Let D be the dense subspace $B \cap H^{2+\alpha}$ of B. We shall also consider the following norms for functions $u(x, t)$ defined on $R^m \times [0, a]$.

$$| u |_a^{2+\alpha} = \sum_{|2i|+|j|\leqslant 2} \| D_t^i D_x^j u \| + \sum_{|2i|+|j|=2} \langle D_t^i D_x^j u\rangle_x^\alpha$$

$$+ \sum_{|j|=1} \langle D_x^j u\rangle_t^{\alpha/2+1/2} + \sum_{|j|=2} \langle D_x^j u\rangle_t^{\alpha/2}$$

$$+ \langle D_t u\rangle_t^{\alpha/2}, \quad \text{and} \quad | u |_a^\alpha = \| u \| + \langle u\rangle_x^\alpha + \langle u\rangle_t^{\alpha/2}.$$

For a discussion of these norms see [12] p. 7.

We consider now the nonlinear elliptic differential operator A on D defined by

$$Au(x) = \sum a_{ij}(x, u(x))\, D^{ij}u(x) + \sum b_i(x, u(x))\, D^i u(x) + c(x, u(x))\, u(x)$$

with the following two assumptions concerning the coefficients.

5.1 *Assumption.* A is uniformly elliptic, which means that there are positive constants p and q such that for each x and y in R^m and u in D

$$p\, | y |^2 \leqslant \sum a_{ij}(x, u(x))\, y_i y_j \leqslant q\, | y |^2.$$

5.2 *Assumption.* The coefficients of A and their u-derivatives of order $\leqslant 2$ are continuous and bounded by a constant C. The coefficients and their u-derivatives are uniformly Hölder continuous in x (exponent α). The coefficients $c(x, u)$ and the u-derivatives of all the coefficients vanish at infinity in x, uniformly in u.

Note. The assumption of vanishing at infinity may be weakened in other choices for the space B. This space was chosen because of the availability of the results in Section 2 which appear in [7]. These apply because B is isomorphic to the space of continuous functions on the one-point compactification of R^m. Improvements of Section 2 are presently under study.

We now proceed to show that the approximation conditions of Section 3 are satisfied.

5.3 LEMMA. *A is a G-differentiable mapping of D into B.*

$$dA(u)(w)(x) = \sum a_{ij}(x, u(x))\, D_x^{ij}\, w(x) + \sum b_i(x, u(x))\, D_x^{\,i} w(x)$$
$$+ d(x, u(x))\, w(x),$$

where

$$d(x, u(x)) = \sum D_u a_{ij}(x, u(x))\, D_x^{ij}\, u(x) + \sum D_u b_i(x, u(x))\, D_x^{\,i} u(x)$$
$$+ D_u c(x, u(x)) \cdot u(x) + c(x, u(x)),$$

and $\|f(u, tw)\| \leqslant Ct^2 (\sum \| wD^k w \| + \| w^2 D^k(u + tw)\|)$.

Proof. Each term of $Au(x)$ is of the form $a(x, u(x))\, D^k u(x)$; hence a term of $A(u + w)(x) - Au(x)$ can be written as

$$a(x, u(x) + w(x))\, D^k u(x) + a(x, u(x) + w(x))\, D^k w(x) - a(x, u(x))\, D^k u(x)$$
$$- a(x, u(x))\, D^k w(x) + a(x, u(x))\, D^k w(x).$$

If $a(x, u + w) - a(x, u) = w\, \partial a/\partial u + \epsilon(u, w)$, then the above term is $a(x, u(x))\, D^k w(x) + w(x)(\partial a/\partial u) D^k u(x) + w(x)(\partial a/\partial u)\, D^k w(x) + \epsilon(u(x), w(x))\, D^k(u(x) + w(x))$. Since $\partial a/\partial u$ is Lipschitz continuous we have $| \epsilon(u, w)| = | \int_0^w (\partial a(u + s)/\partial u - \partial a(u)/\partial u)\, ds | \leqslant C \mid w \mid^2$.

5.4 LEMMA. *The operator $-A$ is K-accretive where for each u, $\| K(u)\| = C(1 + \sum \| D^k u \|)$.*

Proof. Suppose that u and v are in D and choose x such that $| u(x) - v(x)| = \| u - v \| = u(x) - v(x)$. Since $D^1(u - v)(x) = 0$ and the matrices $a_{ij}(x, u(x))$ and $D^{ij}(v - u)(x)$ are positive definite we have

$$Au(x) - Av(x) = L(x) + \sum a_{ij}(x, u(x))\, D^{ij}(u - v)(x) \leqslant L(x)$$

where

$$\begin{aligned} L(x) &= \sum (a_{ij}(x, u(x)) - a_{ij}(x, v(x)))\, D^{ij}v(x) \\ &\quad + \sum (b_i(x, u(x)) - b_i(x, v(x))\, D^i v(x) \\ &\quad + (c(x, u(x)) - c(x, v(x))\, v(x) + c(x, u(x))\,(u - v)(x) \leqslant | L(x)| \\ &\leqslant C\, | u(x) - v(x)|\, (1 + \sum | D^k v(x)|) \leqslant (u(x) - v(x))\, K(v). \end{aligned}$$

It follows that

$$\begin{aligned} \| u - v - rAu + rAv \| &\geqslant u(x) - v(x) - r(Au(x) - Av(x)) \\ &\geqslant (1 - rK(v))\,(u(x) - v(x)) = (1 - rK(v))\, \| u - v \|. \end{aligned}$$

5.5 LEMMA. *Suppose that u is a function from an interval* $[0, a]$ *into* D *such that for each* t, $| u(t)|^{2+\alpha} \leqslant C$ *and* $\langle D_x{}^k u(\cdot)(x)\rangle^{\alpha/2} \leqslant C$ *where* $| k | \leqslant 2$ *and* C *is a positive constant; then there is a fundamental solution* $Z = Z(u, t, s, x, y)$ *for the parabolic equation*

$$D_t w(x, t) = dA(u(t))\, w(x, t). \tag{1}$$

If w_0 *is in* B *and if*

$$w(x, t) = \int Z(u, t, 0, x, y)\, w_0(y)\, dy, \tag{2}$$

then we have the following properties.

(i) *The function* $w(x, t)$ *is the unique solution to equation* (1) *such that* $w(x, 0) = w_0(x)$.

(ii) $\lim_{t\to 0} w(x, t) = w(x, 0)$ *for each* x.

(iii) $| w |_a^{2+\alpha} \leqslant C(u)| w_0 |^{2+\alpha}$ *where* $C(u)$ *is a constant depending on* p, q, α, a, *and the bounds for the coefficients of* $dA(u(t))$

(iv) $| Z(u, t, s, x, y)| \leqslant cg(t - s, x - y)$ *where* $g(r, z) = (4\pi r)^{-n/2} \exp(-c\, | z |^2/4r)$.

Proof. According to well known results concerning second order parabolic equations it suffices to show the following three things (cf. [6], p. 22).

(a) The operator $dA(u(t))$ is uniformly elliptic: This is true by by 5.1 since $u(t)$ is in D.

(b) The coefficients of $dA(u(t))$ are bounded: Since by 5.2 the coefficients of A are bounded, it suffices to obtain a bound for the coefficient $d(x, u(t))$. However, this only requires a bound for the

x-derivatives of $u(t)$ which is given the hypothesis, and a bound for the u-derivatives for the coefficients of A which is given in 5.2.

(c) The coefficients of $dA(u(t))$ are uniformly Hölder continuous in x and t: This is proved by straight forward calculations. The Hölder continuity in x of the coefficients of A and their u-derivatives is given in 5.2 and that of the x-derivatives of $u(t)$ is given by the hypothesis $|u(t)|^{2+\alpha} \leqslant C$. The Hölder continuity in t of the coefficients of A requires the bound for the u-derivatives of the coefficients of A and the Hölder continuity in t of u. The Hölder continuity in t of $d(x, u(t)(x))$ requires bounds for the u-derivatives of the coefficients and for the x-derivatives of $u(t)$, Hölder continuity in t of $u(t)$ and its x-derivatives and the bound for the second u-derivatives of the coefficients of A.

5.6 Lemma. *If u satisfies the hypotheses of Lemma 5.5, if w_0 is in B and $w(x, t)$ is given as in (2), then $w(x, t)$ is in B for each $t > 0$. Hence, if w_0 is in D then $w(\cdot, t)$ is in D.*

Proof. The second statement follows from (iii) above if we know the first statement. Since it is known that $w(x, t)$ is continuous it suffices to show that if $w_0(x) = v_0(x) + c$ where c is a real number and v_0 is continuous and vanishes at infinity, then $w(x, t) = v(x, t) + c$ where for each t, $v(\cdot, t)$ vanishes at infinity. To this end let $L = dA(u(t)) - d(x, u(t)(x))$ and suppose that Z_1 is the fundamental solution to the diffusion equation $D_t w(x, t) = Lw(x, t)$. Then $w(x, t)$ satisfies the equation $D_t w(x, t) - Lw(x, t) = -d(x, u(t)(x))\, w(x, t)$ so that

$$w(x, t) = \int Z_1(u, t, 0, x, y)\, w(y, 0)\, dy - \int_0^t ds$$
$$\times \int Z_1(u, t, s, x, y)\, d(y, u(s)(y))\, w(y, s)\, dy$$

(see [12] p. 389). Since $\int Z_1(u, t, s, x, y)\, dy = 1$, $w(x, t) - c = \int Z_1(u, t, 0, x, y)\, v_0(y)\, dy + \int_0^t ds \int Z_1(u, t, s, x, y)\, d(y, u(s)(y))\, w(y, s)\, dy$. This goes to 0 at infinity because of the following facts: $v_0(y)$ goes to 0 at infinity; since by 5.2 the u-derivatives of the coefficients of A go to 0 at infinity and since the x-derivatives of u are bounded, $d(y, u(s)(y))$ goes to 0 at infinity; $w(y, s)$ is bounded uniformly in s since it is continuous; $|Z_1(u, t, s, x, y)| \leqslant cg(t - s, x - y)$; (see (iv) above) and $\int_{|z| \geqslant r} g(s, z)\, dz \leqslant c/r$ for $s \leqslant t$ and for some constant c.

5.7 Lemma. *If E is a collection of functions from an interval*

$[0, a]$ *into* D, C *is a positive number, and if for each* u *in* E, $|u(t)|^{2+\alpha} \leqslant C$ *and* $\langle D_x^k u(\cdot)(x)\rangle_t^{\alpha/2} \leqslant C$ *then* E *is a generating class for* dA *on* $[0, a]$.

Proof. Referring to Definition 3.4 we see immediately that (a) is satisfied. To show (b) we let $T(u, t, s)\, w(x) = \int Z(u, t, s, x, y)\, w(y)\, dy$ for each w in B where Z is given as in Lemma 5.5. Lemma 5.5 shows that $T(u, t, s)w$ is in $H^{2+\alpha}$ and Lemma 5.6 shows that it is in B, hence in D. The condition (c) follows from (iii) in Lemma 5.5 and the fact that the coefficients of $dA(u(t))$ are uniformly bounded as seen in the proof of Lemma 5.5. Finally, to show (d), since $\| f(u, tw)/t \| \leqslant ct(\sum \| wD^k w \| + \| w^2 D^k(u + tw)\|)$ by Lemma 5.3, we need only show that $w(s + t)D^k w(s + t)$ and $w(s + t)^2 D^k(u(s) + tw(s + t))$ are bounded where $w(s + t) = T(u(s), t, 0)\, T(u, s, 0)w_0$, and w_0 is in D. But this follows from the fact that $|u(t)|^{2+\alpha} \leqslant c$ and the fact that $|w(s + t)|^{2+\alpha} \leqslant c(u)|w_0|^{2+\alpha}$.

In order to verify the approximation conditions as well as to establish the hypothesis of Lemma 2.7, we need more detailed information concerning the dependence of the constant $c(u)$ of Lemma 5.5 (iii) upon the x-derivatives of $u(t)$. We consider a uniformly elliptic linear operator L with bounded, Hölder continuous coefficients given by the formula

$$\begin{aligned} Lw(x, t) &= \sum a_{ij}(x, t)\, D_x^{ij}\, w(x, t) \\ &\quad + \sum b_i(x, t)\, D_x^{i} w(x, t) + c(x, t)\, w(x, t) - d(x, t)\, w(x, t). \end{aligned}$$

Let $w(x, t)$ be a solution to the equation $D_t w(x, t) = Lw(x, t)$, and let $L_1 w(x, t) = Lw(x, t) + d(x, t)$ so that $w(x, t)$ is a solution to the equation $D_t w(x, t) - L_1 w(x, t)w(x, t) = d(x, t)\, w(x, t)$. We shall determine the effect of the coefficient $d(x, t)$. Let $Z_1 = Z_1(t, s, x, y)$ be the fundamental solution associated with the operator L_1 so that (see [12], p. 389)

$$\begin{aligned} w(x, t) &= \int Z_1(t, 0, x, y)\, w(y, 0)\, dy + \int_0^t ds \int Z_1(t, s, x, y)\, d(y, s)\, w(y, s)\, dy \\ &= v_0(x, t) + p_0(x, t). \end{aligned}$$

For each $n \geqslant 1$ define $v_n(x, t) = \int_0^t ds \int Z_1(t, s, x, y)\, d(y, s) v_{n-1}(y, s)\, dy$ and $p_n(x, t) = \int_0^t ds \int Z_1(t, s, x, y)\, d(y, s)\, p_{n-1}(y, s)\, dy$. Since $p_0(x, t) = v_1(x, t) + p_1(x, t)$, we have $p_0(x, t) = \sum_{n=1}^{\infty} v_n(x, t)$. Let $K = \int |Z_1(t, s, x, y)|\, dy$, noting that if $c(x, t) \equiv 0$ then $K = 1$. Let d be a bound for $|d(x, t)|^{\alpha}$.

5.8 LEMMA. *If $w(x, t)$, d, and K are as above then for each x and t, $|w(x, t)| \leqslant K \| w(\cdot, 0)\| \exp(t\,Kd)$.*

Proof. For each $n \geqslant 1$,

$$
\begin{aligned}
|v_n(x, t)| &\leqslant d \int_0^t ds \int |Z_1(t, s, x, y)|\, |v_{n-1}(y, s)|\, dy \\
&\leqslant d^n \int_0^t ds \int_0^s ds_1 \int_0^{s_1},\ldots, \int_0^{s_{n-2}} ds_{n-1} \int |Z_1(t, s, x, y)| \int \ldots \\
&\qquad \times \int |Z_1(s_{n-1}, 0, y_{n-1}, y_n)|\, |w(y_n, 0)|\, dy_n \\
&\leqslant \| w(\cdot, 0)\|\, d^n K^{n+1} \int_0^t ds \int_0^s ds_1 \int_0^{s_1} \ldots \int_0^{s_{n-2}} ds_{n-1} \\
&= \| w(\cdot, 0)\|\, K \cdot K^n d^n t^n / n!
\end{aligned}
$$

5.9 LEMMA. *If $w(x, t)$, d, and K are as above and a is positive, then $|w|_a^{2+\alpha} \leqslant |w(\cdot, 0)|^{2+\alpha} \cdot C_1(1 + d + d^2 \ \exp(dKt))$ where C_1 depends only on L_1.*

Proof. Since $|v_n|_a^{2+\alpha} \leqslant C_1 |d(\cdot, \cdot)\, v_{n-1}(\cdot, \cdot)|_a^{\alpha}$ (see [12] p. 390) and

$$
|w|_a^{2+\alpha} \leqslant \sum_{n=0}^{\infty} |v_n|_a^{2+\alpha} \leqslant C_1 |w(\cdot, 0)|^{2+\alpha} + |v_1|^{2+\alpha} + \sum_{n=2}^{\infty} |v_n|_a^{2+\alpha},
$$

we estimate $|v_{n-1}|_a^{\alpha}$. The Hölder constant for x is determined by the following inequalities.

$$
\begin{aligned}
&|v_{n-1}(x, t) - v_{n-1}(z, t)| \\
&\quad \leqslant d \int_0^t ds \int |Z_1(t, s, x, y) - Z_1(t, s, z, y)| \cdot |v_{n-2}(y, s)|\, dy \\
&\quad \leqslant (\| w(\cdot,0)\| K^{n-1} d^{n-1}/(n-2)!) \int_0^t s^{n-2}\, ds \int |x - z|\, |D_\beta Z_1(t, s, x', y)|\, dy
\end{aligned}
$$

where D_β is the directional derivative in the direction of $x - z$. Since $\int |D_\beta Z_1(t, s, x', y)|\, dy \leqslant \int \sum |D_x{}^i Z_1(t, s, x', y)|\, dy \leqslant mc(t - s)^{-1/2}$, (see [12], p. 376) and $\int_0^t s^{n-2}(t - s)^{-1/2}\, ds = t^{n-3/2} B(n - 1, 3/2) = t^{1/2}\Gamma(3/2)$, it follows that

$$
|v_{n-1}(x, t) - v_{n-1}(z, t)| \leqslant |x - z| \cdot C \| w(\cdot, 0)\| K^{n-1} d^{n-1} t^{n-2} t^{1/2}/(n-2)!.
$$

The Hölder constant for t is obtained as follows. If $r < t$ then

$$
\begin{aligned}
&| v_{n-1}(x, t) - v_{n-1}(x, r)| \\
&\quad \leqslant \int_0^r ds \int | Z_1(t, s, x, y) - Z_1(r, s, x, y)| \cdot | d(y, s)\, v_{n-1}(y, s)|\, dy \\
&\qquad + \left| \int_r^t ds \int | Z_1(t, s, x, y)\, d(y, s)\, v_{n-2}(y, s)|\, dy \right. \\
&\quad \leqslant [d \,\| w(0, \cdot)\|\, K^{n-1} d^{n-2}/(n-2)!] \left[\int_0^r s^{n-2}\, ds \int |(t - r)\, D_t Z(t', s, x, y)|\, dy \right. \\
&\qquad \left. \times \int_r^t s^{n-2}\, ds \int Z_1(t, s, x, y)\, dy \right] \\
&\quad \leqslant | t - r |^{1/2} \cdot C \,\| w(\cdot, 0)\|\, K^{n-1} d^{n-1} t^{n-2}/(n-2)!.
\end{aligned}
$$

Since $| v_{n-1}(x, t)|$ is estimated as in Lemma 5.8, we have

$$| v_{n-1} |_a^\alpha \leqslant C \cdot Kd \,\| w(\cdot, 0)\|\, K^{n-2} d^{n-2} t^{n-2}/(n-2)!.$$

Since

$$| v_1 |_a^{2+\alpha} \leqslant C_1 \,| d(\cdot, \cdot)\, v_0(\cdot, \cdot)|_a^\alpha \leqslant C_1 d \,| v_0 |_a^\alpha \leqslant C_1 d \,| v_0 |_a^{2+\alpha} \leqslant C_2 d \,| w(\cdot, 0)|^{2+\alpha}$$

the conclusion is evident.

5.10 Corollary. *If u maps $[0, a]$ into D, $| u(t)|^{2+\alpha} \leqslant C$ and $\langle D_x{}^k u(t)\rangle_t^{\alpha/2} \leqslant C$ and w is the solution to the equation $D_t w(x, t) = dA(u(t))\, w(x, t)$ on the interval $[0, a]$ then*

$$| w |_a^{2+\alpha} \leqslant C_1 \,| w(\cdot, 0)|^{2+\alpha} (1 + q + q^2 \exp(C_2\, qa))$$

where $q = \sup | u(t)|^{2+\alpha}$ and C_1 and C_2 are constants dependent upon the characteristics of the coefficients of A and not dependent upon $u(t)$.

5.11 Theorem. *If A is the nonlinear operator on D given at the beginning of this section, if A satisfies 5.1 and 5.2 and if $A\, u_0$ is in D, then A and u_0 satisfy the approximation conditions on some interval $[0, a]$, hence the conclusion of Theorems 3.12 and 3.13 hold.*

Proof. Let b be a positive number and let $f(y) = C_1 \,| Au_0 |^{2+\alpha} (1 + y + y^2 \exp(C_2 yb))$ where C_1 and C_2 are as in Corollary 5.10. Let $y_0 = | u_0 |^{2+\alpha}$. Since f is Lipschitz continuous, there is a unique solution $y(t)$ to the differential equation $y'(t) = f(y(t))$ on some interval $[0, c]$ and with $y(0) = y_0$. Let $a = \min\{b, c\}$. Since y_0 is positive and $f(y)$ is positive and increasing for positive y, the function

$y(t)$ is increasing. We now show that $y(t)$ is an upper bound for the functions $|u_n(t)|^{2+\alpha}$ where $u_n(t)$ is given as in Definition 3.9. For this it suffices to show that if $d(k) = \sup_{t \leqslant kr} |u_n(t)|^{2+\alpha} \leqslant y(kr)$ then $|u_n(kr+s)|^{2+\alpha} \leqslant y(kr+s)$. By Corollary 5.10,

$$|T(u_n^{k-1}(kr), s)\, T(u_n^{k-1}, kr, 0)\, Au_0|_{kr+t}^{2+\alpha}$$
$$\leqslant C_1 |Au_0|^{2+\alpha} (1 + d(k) + d(k)^2 \exp(C_2 d(k) b) = f(d(k)),$$

so $|u_n(kr+s)|^{2+\alpha} \leqslant y(kr) + sf(y(kr))$. If $y(kr) + sf(y(kr)) > y(kr+s)$, then $sf(y(kr)) > y(kr+s) - y(kr) = sy'(kr+s') = sf(y(kr+s')) > sf(y(kr))$, which is impossible. Therefore $|u_n(kr+s)|^{2+\alpha} \leqslant y(kr) + sf(y(kr)) \leqslant y(kr+s)$. It follows that $y(a)$ is an upper bound for $|u_n(t)|^{2+\alpha}$ independent of n and t. This shows also that there is a uniform upper bound for $\| Ku_n(t))\|$. It remains to show that $D^j u_n(t)$ is uniformly Hölder continuous in t, which comes from the following inequalities.

$$\| D^j u_n(kr+t) - D^j u_n(kr+s)\|$$
$$= \|(tD^jT(u_n(kr), t) - sD^jT(u_n(kr), t)$$
$$+ sD^jT(u_n(kr), t) - sD^jT(u_n(kr), s))\, T(u_n, kr, 0)\, Au_0\|$$
$$\leqslant \|(t-s)\, D^jT(u_n(kr), t)\, T(u_n, kr, 0)\, Au_0\| + \| sD^j(T(u_n(kr), t)$$
$$- T(u_n(kr), s))\, T(u_n, kr, 0)\, Au_0\|.$$

5.12 Lemma. *If E is a collection of functions from an interval $[0, a]$ into D such that $|u|^{2+\alpha} \leqslant C$ and $\langle D_x{}^j u\rangle_t^{\alpha/2} \leqslant C$ for each u in E, $k \leqslant 2$, $0 < \alpha < 1$, where C is a positive constant, then there are positive numbers M and c such that for each u in E, the transition operator $T(u, t, s)$ as determined in Lemma 5.7 satisfies condition $P(M, c)$.*

Proof. As in Lemma 5.6 let $L = dA(u(t)) - d(x, u(t)(x))$ where u is given in E, and let Z_1 be the fundamental solution to the equation $D_t w = Lw$. We have $\int Z_1(u, t, s, x, y)\, dy = 1$ so that the transition operator P defined by $P(t, s)\, w(x) = \int Z_1(u, t, s, x, y)\; w(y)\, dy$ is a Markov transition operator. The condition $|Z_1(u, t, s, x, y)| \leqslant Mg(t, s, x, y)$ means that $|P(t, s, x, dy)| \leqslant Mg(t, s, x, y)$. Finally, as in the comments preceeding Lemma 5.8, we see that for w_0 in B, $w(x, t) = T(u, t, s)\, w_0(x) = \int T(u, t, s, x, dy)\, w_0(y) = \sum_{n=0}^{\infty} v_n(x, t)$ where $v_0(x, t) = \int Z_1(u, t, s, x, y)\, w_0(y)\, dy$ and $v_{n+1}(x, t) = \int_s^t dr \int Z_1(u, t, r, x, y)\, d(y, u(r)(y))\; v_n(y, r)\, dy$. Suppose that C is an upper bound for $|d(y, u(r)(y))|$ independent of u, y, and r. Proceeding as in Lemma 5.8 but with $Z_1 \geqslant 0$ we have

$$v_n(x, t) \leqslant C^n \int_s^t dr \int_s^r dr_1 \int_s^{r_1} \cdots \int_s^{r_{n-2}} dr_{n-1} \int Z_1(u, t, r, x, y) \int \cdots$$

$$\times \int Z_1(u, r_{n-1}, s, y_{n-1}, y_n) | w_0(y_n)| \, dy_n$$

$$= C^n \int_s^t dr \int_s^r \cdots \int_s^{r_{n-2}} dr_{n-1} \int Z_1(u, t, s, x, y_n) \mid w_0(y_n)| \, dy_n$$

$$= (C^n(t-s)^n/n!) \, P(t, s) \mid w_0 \mid (x),$$

so that $w(x, t) \leqslant P(t, s)| \, w_0 \, | \, (x) \exp(c(t-s))$. This suffices to show that $| \, T(t, s, x, dy)| \leqslant P(t, s, x, dy) \exp(c(t-s))$.

We summarize with the following theorem.

5.13 THEOREM. *If A is the second order elliptic partial differential operator from D into B satisfying the assumptions* 5.1 *and* 5.2; *if u_0 is in D and Au_0 is in D, then there is an interval* $[0, a]$ *such that*

(1) *There is a unique mapping u from* $[0, a]$ *into D such that $D_t u(t) = Au(t)$. The functions u_n of Section* 3 *converge to u.*

(2) *For each x_0 in R^m there are Radon measures $m(n, t)$ and $m(t)$ on $C([0, a], R^m)$ such that $Au(t)(x_0) = \int Au_0(w(0))m(t)(dw)$, and a subsequence of $w_n(t) = \int Au_0(w(0))m(n, t)(dw)$ converges to $Au(t)(x_0)$. If $E = \{w: w(t) \neq x_0\}$ then $m(t)(E) = m(n, t)(E) = 0$.*

(3) *The measures $m(n, t)$ and $m(t)$ determine signed processes $x(n, t)$ and $x(t)$ which have continuous trajectories beginning at the point x_0 at time t and running (backwards) until time* 0.

Proof. By Theorems 5.11 and 3.13 there is an interval $[0, a]$ and a function u from $[0, a]$ into B such that the approximations $u_n(t)$ converge to $u(t)$ in B uniformly in t, and there is a subsequence of the sequence $Au_n(t)$ which converges to a continuous function $v(t)$ such that $u(t) = u_0 + \int_0^t v(s) \, ds$. Since, as in the proof of Theorem 5.11, there is a constant C such that $| \, u_n(t)|^{2+\alpha} \leqslant C$ and $\langle D_x{}^k u_n(t)(x)\rangle_t^{\alpha/2} \leqslant C$, the sequences $D_x{}^k u_n$(for $| \, k \, | \leqslant 2$) of functions from $[0, a]$ into B are uniformly bounded and uniformly equicontinuous. Hence by Ascoli's theorem there are subsequences converging to uniformly Hölder continuous functions p^k from $[0, a]$ into B. Since the sequence $u_n(t)$ converges to u and since the derivatives $D_x{}^k u_n(t)$ are uniformly Hölder continuous in x, it follows that $p^k(t) = D_x{}^k u(t)$ for each t.

Thus there is a subsequence u_{n_j} such that for each k, $D_x{}^k u_{n_j}(t)$ converges to $D^k u(t)$. It follows that $Au_{n_j}(t)$ converges to $Au(t)$ so that $Au(t) = v(t)$. Since v is continuous and $u(t) = u_0 + \int_0^t v(s)\, ds$ we have $D_t u(t) = v(t) = Au(t)$. This proves (1). The statement (2) follows from Lemma 5.12 and Theorems 4.2 and 4.3. The statement (3) is a restatement of parts of (1) and (2) in the language of Section 2.

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